DCT-003-2013008 Seat No. ________

B. Sc. (Sem. III) (W.E.F. 2019) (CBCS) Examination

August – 2022

Mathematics : Paper-03(A)

(Linear Algebra & Real Analysis)

Faculty Code : 003
Subject Code : 2013008

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Attempt any five questions.
(2) Figures to the right indicates full marks of the question.

1 (A) Answer the following questions : **4**

(1) Define: Linear span of a set.

(2) The set $A = \{e^x, e^{2x}\}$ is LI or LD.

(3) Every nonempty subset of a linearly independent set is linearly independent. (True/False)

(4) The sum of two subspace is subspace. (True/False)

(B) Prove that intersection of two subspaces of a vector space is also a vector space. **2**

(C) Check whether the set $\{(1, 2, 4), (2, 1, 3), (0, 1, 2)\} \subset \mathbb{R}^3$ is LI or not. **3**

(D) Prove that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$ with the following operations. **5**

For $(a, b), (c, d) \in \mathbb{R}^2, (a, b) + (c, d) = (a + c, b + d)$

For $(a, b) \in \mathbb{R}^2,\ \alpha \in \mathbb{R}\ \alpha (a, b) = (\alpha a, \alpha b)$.

2 (A) Answer the following questions : **4**

(1) Define: Linearly independent set.

(2) The set A = {cos x, sin x} is LI or LD.

(3) Let $(V, +, \cdot)$ be a vertical space over a field F. For any $\alpha \in F$, $\alpha 0 =$ ________ .

(4) In any vector space, the opposite vector of any vector is always unique. (True/False)

(B) Check whether $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $M_2(\mathbb{R})$ or not. **2**

(C) Prove that a nonempty subset W is a subspace of a vector space V if and only if $\forall w_1, w_2 \in W$ and $\alpha, \beta \in \mathbb{R}$, $\alpha w_1 + \beta w_2 \in \mathbb{R}$. **3**

(D) Let $A = \{\overline{v}_1, \overline{v}_2, \ldots.., \overline{v}_n\}$ be a subset of a vector space V over $\mathbb{R}$. If $\overline{v}_k \in A (1 \le k \le n)$ is linear combination of remaining vectors $\overline{v}_1, \overline{v}_2, \ldots., \overline{v}_{k-1}, \overline{v}_{k+1}, \ldots., \overline{v}_n$, then prove that span $(\{\overline{v}_1, \overline{v}_2, \ldots., \overline{v}_n\})$ = span $(\{\overline{v}_1, \overline{v}_2, \ldots., \overline{v}_{k-1}, \overline{v}_{k+1}, \cdot, \overline{v}_n\})$. **5**

3 (A) Answer the following questions : **4**

(1) What is the coordinate vector of (1, 0) relative to the basis {(1, 1), (0, 1)}?

(2) For a vector space V if $U \oplus W = V$ then $U \cap W =$ ________ .

(3) What is dimension of $M_2(\mathbb{R})$?

(4) Define: Basis of a vector space.

(B) If vectors (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are LD, then show that **2**

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

(C) Extend set A = {(1, 1, 1), (2, 0, 0)} of vector space $\mathbb{R}^3$ to form a basis of $\mathbb{R}^3$. **3**

(D) Let $W_1 = \{(x-y,\ y+z,\ y,z) : x,y,z \in \mathbb{R}\}$, **5**

$W_2 = \{(x,\ x+y,\ x+y+z,\ y-z) : x,y,z \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^4$. Find $dim W1$, $dim(W_1 + W_2)$, $dim(W_1 \cap W_2)$.

4 (A) Answer the following questions : **4**

(1) Define: Complementary subspace.

(2) Define: Dimension of a vector space.

(3) The dimension of polynomial space $P_n(\mathbb{R})$ is *n*. (True/False)

(4) Every linearly independent set of vector space can be extended to form a basis. (True/False)

(B) If *W* is a subspace of finite dimensional vector space *V*. Show that $\dim W \leq \dim V$. **2**

(C) Extend set $A = \{1-x+x^2, 2x-x^2+x^3\}$ of vector space $P_3(\mathbb{R})$ to form a basis of $P_3(\mathbb{R})$. **3**

(D) If *U* and *W* are two subspaces of finite dimensional vector space *V*, then prove that $dim(U+W) = dim U + dim W - dim\ (U \cap W)$. **5**

5 (A) Answer the following questions : **4**

(1) Define: Gradient.

(2) Find the divergence of $\overline{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$.

(3) If $\overline{a}$ is a constant vector, then curl$\overline{a}$ = ______.

(4) If $\nabla \cdot \overline{F} = div\overline{F} = 0$, then vector field $\overline{F}$ is solenoidal. (True/False)

(B) Prove that $\nabla(fg) = g\nabla f + f\nabla g$, where f and g are scalar function. **2**

(C) Find the curl of $\overline{F}(x,y,z) = x^2yz\,\hat{i} + xy^2\,\hat{j} + xyz^2\,\hat{k}$ at $(1,\ 2,\ 3)$. **3**

(D) Prove that, **5**

(i) $div(\overline{r} \times \overline{a}) = 0$.

(ii) $curl(\overline{r} \times \overline{a}) = -2\overline{a}$, where $\overline{a}$ is a constant vector.

6 (A) Answer the following questions : **4**

(1) Define: Divergence.

(2) If $\overline{a}$ is a constant vector, then $div(\overline{a}) =$ _______.

(3) Find the gradient of scalar function $\phi(x,y,z) = x^2 + y^2 + z^2$.

(4) If $\nabla \times \overline{F} = curl\overline{F} = 0$ then vector field $\overline{F}$ is irrotational. (True/False)

(B) If $\overline{F} = (x^3z - axyz)\,\hat{i} + (xy - 3x^2yz)\,\hat{j} + (yz^2 - xz)\,\hat{k}$ is a solenoidal. Then find the value of a. **2**

(C) Prove that the function $H = e^{-\lambda x}(C_1 \sin\lambda y + C_2 \cos\lambda y)$ satisfies the Laplace Equation, where λ, C_1, C_2 are constants. **3**

(D) If $\overline{F} = (x^3, y^3, z^3)$ then prove that $curl\overline{F} = \overline{0}$ and $grad(div\overline{F}) = 6\overline{r}$. **5**

7 (A) Answer the following questions : **4**

(1) Evaluate $\int_0^2 \int_0^2 x\,dx\,dy$.

(2) Evaluate $\int_0^1 \int_0^1 \int_0^1 x^2\,dx\,dy\,dz$.

(3) Change the order of integration $\int_0^4 \int_y^4 f(x,\ y)\,dx\,dy$.

(4) Evaluate: $\int_{-1}^1 \int_0^x dy\,dx$

(B) Change $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dydx$ into polar coordinates. **2**

(C) Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} r\sin\theta\, drd\theta$ **3**

(D) Prove that volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4}{3}\pi abc$. **5**

8 (A) Answer the following questions : **4**

(1) Evaluate $\int_0^1 \int_0^x y\,dydx$.

(2) Evaluate $\int_0^1 \int_0^1 \int_0^1 xy\,dxdydz$.

(3) Change the order of integration $\int_0^4 \int_0^x f(x, y)\,dydx$.

(4) Evaluate $\int_0^1 \int_0^1 x^{x+y}\,dydx$.

(B) Evaluate $\int_0^a \int_0^b \int_0^c (x+y+z)\,dxdydz$. **2**

(C) $\int_0^1 \int_0^{x^2} \int_0^{x+y} (2x-y-z)\,dzdydx$. **3**

(D) Prove that $\iint_R (x+y)^2 dxdy = \frac{255}{4}$, where R is region bounded by lines $x+y=4, x+y=1, x-2y=-2$ and $x-2y=1$. **5**

9 (A) Answer the following questions: **4**

(1) State Divergence theorem.

(2) Evaluate $\int_{(0,0)}^{(2,2)} y^2 dx$.

(3) Define Beta function.

(4) $\Gamma\left(\frac{1}{2}\right)$ = ________.

(B) Prove that $\int_0^{\infty} e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$, $a > 0, n > 0$. **2**

(C) Evaluate $\int_0^{\infty} \frac{x^4}{(1+x^2)^4} dx$. **3**

(D) State and prove Green's Theorem. **5**

10 (A) Answer of following questions : **4**

(1) State Stoke's Theorem.

(2) Evaluate $\int_{(2,1)}^{(1,2)} y dx$.

(3) $\beta(2,3)$ = ________.

(4) $\Gamma\left(\frac{3}{2}\right)$ = ________.

(B) Evaluate $\iint\limits_S x^2 dydz + y^2 dxdz + 2z(xy - x - y)dxdy,$ **2**

where $S : 0 \le x, y, z \le 1$ a solid surface.

(C) Verify the Green's theorem for $f(x, y) = e^{-x} \sin y,$ **3**

$g(x, y) = e^{-x} \cos y$ and C is the square with vertices

at $(0,0), \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right).$

(D) State and prove relation between beta and gamma function. **5**